Quantum superoscillator algebra for OSp(2/2)



1993 J. Phys. A: Math. Gen. 26 L1085

(http://iopscience.iop.org/0305-4470/26/20/005)

J. Phys. A: Math. Gen. 26 (1993) L1085–L1091. Printed in the UK

LETTER TO THE EDITOR

# Quantum superoscillator algebra for OSp(2/2)

Preeti Parashar† and S K Soni‡

† Department of Physics and Astrophysics, University of Delhi, Delhi 110007, India
‡ Physics Department, SGTB Khalsa College, University of Delhi, Delhi 110007, India

Received 1 July 1993

**Abstract.** Using the $\hat{R}$ matrix formalism we construct a system of $q$-superoscillators, bosonic as well as fermionic, covariant under the coaction of the quantum orthosymplectic supergroup $OSp_{p,q}(2/2)$.

A lot of interest has been generated in the theory of quantum deformations of Lie groups and Lie algebras. The study of $q$-deformations has been further extended to supergroups and superalgebras. In an interesting development, independently of the original formal considerations [1–3], a system of $q$-oscillators has been proposed for $q$-oscillator realizations of quantum Lie algebras and superalgebras. In particular, Chaichian *et al* [4] have succeeded in constructing the algebra of $q$-oscillators (bosonic as well as fermionic) covariant under the coaction of the quantum supergroup $SU_q(n/m)$. Their considerations also allow one to define a system of $q$-oscillators covariant with respect to other supergroups. The purpose of this note is to give a corresponding system of superoscillators covariant with respect to two parameter deformation of the quantum orthosymplectic group OSp(2/2).

The system of deformed creation and annihilation operators can be identified with the coordinates and derivatives of non-commutative spaces covariant under the action of quantum groups [5–7]. A non-commutative differential calculus on a general quadratic algebra covariant under the coaction of arbitrary quantum group has been developed by Wess and Zumino [7]. They assumed the following commutation relations between variables and derivatives:

$$x^i x^j = B^{ij}_{kl} x^k x^l \tag{1.1}$$

$$\partial_j x^i = \delta^i_j + C^{ik}_{jl} x^l \partial_k \tag{1.2}$$

$$\partial_i \partial_j = F^{kl}_{ji} \partial_l \partial_k \tag{1.3}$$

where the matrices $B$, $C$ and $F$ are related to the $\hat{R}$ matrix of the quantum group. Wess and Zumino solved the constraints for these matrices in terms of the $\hat{R}$ matrix. However, Hlavaty [8] has shown that there exist more general solutions of the constraints satisfied by $B$, $C$ and $F$ matrices. The minimal polynomial of the matrix $\hat{R}$ is of degree $m$ and is defined as

$$M(\hat{R}) \equiv (\hat{R}-\lambda_1)(\hat{R}-\lambda_2)\ldots(\hat{R}-\lambda_m)=0 \qquad (m \leqslant n^2). \tag{1.4}$$

0305-4470/93/201085+07 \$07.50  

The roots of the polynomial $M(\hat{R})$ correspond to the eigenvalues of $\hat{R}$ but their multiplicities may differ. The solutions to the constraints are given by the following matrices

$$C=C^k(\hat{R})=\frac{\hat{R}}{\lambda_k} \qquad B=B_k(\hat{R})=E-\frac{M_k(\hat{R})}{K_k} \qquad F=B \tag{1.5}$$

where

$$M_k(\hat{R})=(\hat{R}-\lambda_k)^{-1}M(R) \qquad K_k=\prod_{j\neq k}(-\lambda_j) \qquad k=1,\ldots,n. \tag{1.6}$$

In this way we get a nice construction of quantum hyperplanes

$$Q_k(\hat{R})=C\langle x^1,\ldots,x^n\rangle/\frac{M_k(\hat{R})}{K_k} \tag{1.7}$$

and their differential calculi (given by $B$, $C$, $F$). *The* $m=2$ case has been discussed in the literature quite often. We are here interested in the case when the minimal polynomial is of degree greater than two. The construction, however, works only for the hyperplanes given by $Q_k(\hat{R})$. Differential calculi on other possible hyperplanes cannot be defined. The quantum superplane covariant under the action of GL(1/1) is discussed. Then we extend these considerations to quantum supergroup OSp(2/2) and formulate a consistent differential calculus which is realized in terms of quantum superoscillators.

We start with the following $\hat{R}$ matrix [4, 8–10] which gives one of the eight vertex solutions of the Yang–Baxter equation (YBE), the complete list of which has been given by Hlavaty [8]:

$$\hat{R}=\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \omega & \frac{q}{p} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{p} \end{pmatrix} \tag{2.1}$$

where $p$ and $q$ are two complex deformation parameters and $\omega=q-1/p$. We also take $pq=r^2$. This matrix which is a solution of the braid Yang–Baxter equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}=\hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{2.2}$$

is a 'non-standard' [11] $\hat{R}$-matrix because in the classical limit this reduces not to the usual permutation matrix but to the super (or graded) permutation matrix acting upon a superspace whose first coordinate is bosonic and the second is fermionic:

$$\hat{R}_{q\to 1}\to\mathcal{P}\equiv\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.3}$$

This $\hat{R}$-matrix is related to the $R$-matrix for $q$-deformation of coordinate functions on the supergroup GL(1/1) in the following way

$$\hat{R}=\mathcal{P}R. \tag{2.4}$$

Given a matrix $\hat{R}$ satisfying the braid YBE (given by (2.2)), the matrix $R$ obtained from it by left multiplication with $\mathcal{P}$ can be shown to satisfy the graded Yang–Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12} \tag{2.5}$$

where [9]

$$\begin{aligned}(R_{12})^{i_1i_2i_3}_{j_1j_2j_3}&=R^{i_1i_2}_{j_1j_2}\delta^{i_2}_{j_2}\\ (R_{13})^{i_1i_2i_3}_{j_1j_2j_3}&=(-1)^{i_2(i_3+j_3)}R^{i_1i_3}_{j_1j_3}\delta^{i_2}_{j_2}\\ (R_{23})^{i_1i_2i_3}_{j_1j_2j_3}&=(-1)^{i_1(i_2+i_3+j_2+j_3)}R^{i_2i_3}_{j_2j_3}\delta^{i_1}_{j_1}.\end{aligned} \tag{2.6}$$

This conclusion holds if, and only if, the sum of the degrees of the upper pair of indices of the $\hat{R}$ matrix matches that of the lower pair. This technical requirement is evidently satisfied for the matrix given in (2.1). Throughout this paper it is understood that in exponents like $(-1)^i$ $i$ stands for the $Z_2$-grade of the index $i$, which is 0 for the first row or column of a $2\times2$ matrix and 1 for the second.

The quantum supergroup commutation relations can be written in matrix form as

$$RT_1T_2=T_2T_1R \tag{2.7}$$

where we use the tensoring convention of [9]

$$\begin{aligned}(T_1)^{ab}_{cd}&=(T\otimes I)^{ab}_{cd}=(-1)^{c(b+d)}T^a_c\delta^b_d\\ (T_2)^{ab}_{cd}&=(I\otimes T)^{ab}_{cd}=(-1)^{a(b+d)}T^b_d\delta^a_c.\end{aligned} \tag{2.8}$$

In terms of the $\hat{R}$ matrix (2.7) can be rewritten as

$$\hat{R}T_1T_2'=T_1T_2'\hat{R} \tag{2.9}$$

where

$$T_2'=\mathcal{P}T_1\mathcal{P}. \tag{2.10}$$

Equations (2.2) and (2.9) are the crucial ingredient of FRT formalism [3] in the supersymmetic case [11]. In terms of generators $T^i_j$ equation (2.7) can be recast as

$$R^{ij}_{mp}T^m_kT^p_l=T^j_pT^i_mR^{mp}_{kl}. \tag{2.11}$$

In the present case (2.11) leads to the familiar two-parameter deformation of GL(1/1) [9].

The minimal polynomial of the matrix $\hat{R}$ given by (2.1) is of the second order and

$$\lambda_1=q \qquad \lambda_2=-1/p. \tag{2.12}$$

The relations defining the quantum superplane and the differential calculus corresponding to $B_2(\hat{R})$ are given by

$$x\theta=q\theta x \qquad \theta^2=0 \tag{2.13}$$

$$\begin{aligned}&\partial_x x=1+r^2x\partial_x+(r^2-1)\theta\partial_\theta \qquad \partial_x\theta=p\theta\partial_x\\ &\partial_\theta x=qx\partial_\theta \qquad \partial_\theta\theta=1-\theta\partial_\theta\end{aligned} \tag{2.14}$$

$$\partial_x\partial_\theta=p^{-1}\partial_\theta\partial_x \qquad \partial_\theta\partial_\theta=0. \tag{2.15}$$

Since $\hat{R}(q,p)^{-1}=\mathcal{P}\hat{R}(q^{-1},p^{-1})\mathcal{P}$ the formulae for the alternate differential calculus can be obtained by $q\to1/q$ and $p\to1/p$ and taking the reverse of the factors in products.

The $\hat{R}$ matrix given in (2.1) is closely related to the $\hat{R}$ matrix for the orthosymplectic phase space [12] covariant with respect to quantum OSp(2/2). This phase space is defined as the free associative algebra of the phase space variables

$$X^1 = \hbar\partial_\theta \qquad X^2 = -\mathrm{i}\hbar\partial_x \qquad X^3 = x \qquad X^4 = \theta \tag{2.16}$$

subject to the following commutation relations

$$\begin{aligned}
&X^1X^2 = pX^2X^1 \qquad X^1X^3 = qX^3X^1 \\
&X^2X^4 = pX^4X^2 \qquad X^3X^4 = qX^4X^3 \\
&(X^1)^2 = 0 = (X^4)^2 \\
&X^4X^1 = \hbar - X^1X^4 \\
&X^3X^2 = \mathrm{i}\hbar + r^{-2}X^2X^3 - \mathrm{i}r^{-2}(r^2-1)X^1X^4.
\end{aligned} \tag{2.17}$$

In addition we also have the orthosymplectic constraint

$$\mathrm{i}rX^4X^1 - rX^3X^2 + r^{-1}X^2X^3 + \mathrm{i}r^{-1}X^1X^4 = 0. \tag{2.18}$$

It can be shown that the orthosymplectic $\hat{R}$ matrix solves the braid QYBE. This fact can be used to define four-dimensional quantum superplanes and differential calculi on them. Next we present one concrete example of such a superspace and its differential calculus.

Consider the simplest non-trivial case of a $(2+2)$–dimensional $q$-deformed orthosymplectic phase space which can be regarded as a graduation of the underlying quantum vector superspace involving one bosonic and one fermionic coordinate. The non-vanishing coefficients of the $\hat{R}$ matrix for OSp(2/2) are [12]:

$$\begin{aligned}
&\hat{R}^{11}_{11} = \hat{R}^{44}_{44} = -r^{-1} \qquad \hat{R}^{22}_{22} = \hat{R}^{33}_{33} = r \\
&\hat{R}^{12}_{21} = \hat{R}^{31}_{13} = \hat{R}^{24}_{42} = \hat{R}^{43}_{34} = rq^{-1} \\
&\hat{R}^{21}_{12} = \hat{R}^{13}_{31} = \hat{R}^{42}_{24} = \hat{R}^{34}_{43} = r^{-1}q \\
&\hat{R}^{12}_{12} = \hat{R}^{13}_{13} = \hat{R}^{24}_{24} = \hat{R}^{34}_{34} = r - r^{-1} \\
&\hat{R}^{14}_{14} = (r-r^{-1})(1-r^{-2}) \qquad \hat{R}^{23}_{23} = (r-r^{-1})(1+r^{-2}) \\
&\hat{R}^{32}_{23} = \hat{R}^{23}_{32} = r^{-1} \qquad \hat{R}^{41}_{14} = \hat{R}^{14}_{41} = -r \\
&\hat{R}^{23}_{14} = \hat{R}^{14}_{23} = \mathrm{i}r^{-2}(r-r^{-1}) \qquad \hat{R}^{14}_{32} = \hat{R}^{32}_{14} = -\mathrm{i}(r-r^{-1}).
\end{aligned} \tag{3.1}$$

This $\hat{R}$ matrix satisfies the braid quantum Yang–Baxter equation (2.2). The minimal polynomial for $\hat{R}$ is of the third order

$$M(\hat{R}) \equiv (\hat{R} - rI)(\hat{R} + r^{-1}I)(\hat{R} + r^{-1}I) = 0 \tag{3.2}$$

with roots

$$\lambda_1 = r \qquad \lambda_2 = \lambda_3 = -r^{-1} \tag{3.3}$$

and unit matrix $I$. The root $-r^{-1}$ is a double root. In arriving at the result (3.2) we worked out the components of the $\Omega$ matrix given by

$$\Omega=(\hat{R}-rI)(\hat{R}+r^{-1}I). \tag{3.4}$$

The non-vanishing coefficients of this matrix which turns out to be nilpotent are:

$$\begin{aligned}
&\Omega^{14}_{14}=r^{-2}(1-r^{-2}) \qquad \Omega^{14}_{23}=\mathrm{i}r^{-2}(1-r^{-2}) \qquad \Omega^{14}_{32}=\mathrm{i}(1-r^{-2})\\
&\Omega^{14}_{41}=(1-r^{-2}) \qquad \Omega^{23}_{14}=\mathrm{i}r^{-2}(1-r^{-2}) \qquad \Omega^{23}_{23}=-r^{-2}(1-r^{-2})\\
&\Omega^{23}_{32}=(1-r^{-2}) \qquad \Omega^{23}_{41}=-\mathrm{i}(1-r^{-2}) \qquad \Omega^{32}_{14}=\mathrm{i}r^{-2}(r^2-1)\\
&\Omega^{32}_{23}=r^{-2}(r^2-1) \qquad \Omega^{32}_{32}=-(r^2-1) \qquad \Omega^{32}_{41}=\mathrm{i}(r^2-1)\\
&\Omega^{41}_{14}=r^{-2}(r^2-1) \qquad \Omega^{41}_{23}=\mathrm{i}r^{-2}(r^2-1) \qquad \Omega^{41}_{32}=\mathrm{i}(r^2-1)\\
&\Omega^{41}_{41}=(r^2-1).
\end{aligned} \tag{3.5}$$

The relations defining the quantum hyperplane $Q_1(\hat{R})$ are

$$P^{(1)}(X\otimes X)=0 \tag{3.6}$$

where

$$P^{(1)}\propto(\hat{R}+r^{-1}I)(\hat{R}+r^{-1}I) \tag{3.7}$$

is the projection operator onto the subspace of eigenvectors of $\hat{R}$ with eigenvalue $r$. Explicitly

$$\begin{aligned}
&X^1X^2=-q^{-1}X^2X^1 \qquad X^1X^3=-p^{-1}X^3X^1\\
&X^2X^4=-q^{-1}X^4X^2 \qquad X^3X^4=-p^{-1}X^4X^3\\
&(X^2)^2=0=(X^3)^2\\
&-(r^2+r^{-2})X^1X^4-\mathrm{i}r^{-2}(r^2-1)X^2X^3+\mathrm{i}(r^2-1)X^3X^2+(r^2+r^{-2})X^4X^1=0.
\end{aligned} \tag{3.8}$$

This algebra is dual to the coordinate algebra (2.17) but bigger in size. On the other hand, the defining relations for $Q_2(\hat{R})$ namely, $P^{(2)}(X\otimes X)=0$, reduces to the orthosymplectic constraint (2.18). Here $P^{(2)}$ is the projector onto the subspace of eigenvectors of $\hat{R}$ with eigenvalue $-r^{-1}$.

Making use of the formulae given in section 1 we can obtain the differential calculus on the orthosymplectic superplane (3.8):

$$
\begin{aligned}
&\partial_1 X^1 = 1 + r^{-2}X^1\partial_1 - (1-r^{-2})X^2\partial_2 - (1-r^{-2})X^3\partial_3 - (1-r^{-2})^2 X^4\partial_4,\\
&\partial_1 X^2 = -p^{-1}X^2\partial_1 - \mathrm{i}r^{-2}(1-r^{-2})X^4\partial_3\\
&\partial_1 X^3 = -q^{-1}X^3\partial_1 + \mathrm{i}(1-r^{-2})X^4\partial_2\\
&\partial_1 X^4 = X^4\partial_1\\
&\partial_2 X^1 = -q^{-1}X^1\partial_2 - \mathrm{i}r^{-2}(1-r^{-2})X^3\partial_4\\
&\partial_2 X^2 = 1 - X^2\partial_2 - (1-r^{-2})(1+r^{-2})X^3\partial_3 - (1-r^{-2})X^4\partial_4\\
&\partial_2 X^3 = -r^{-2}X^3\partial_2\\
&\partial_2 X^4 = -p^{-1}X^4\partial_2\\
&\partial_3 X^1 = -p^{-1}X^1\partial_3 + \mathrm{i}(1-r^{-2})X^2\partial_4\\
&\partial_3 X^2 = -r^{-2}X^2\partial_3\\
&\partial_3 X^3 = 1 - X^3\partial_3 - (1-r^{-2})X^4\partial_4\\
&\partial_3 X^4 = -q^{-1}X^4\partial_3\\
&\partial_4 X^1 = X^1\partial_4\\
&\partial_4 X^2 = -q^{-1}X^2\partial_4\\
&\partial_4 X^3 = -p^{-1}X^3\partial_4\\
&\partial_4 X^4 = 1 + r^{-2}X^4\partial_4.
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&\partial_1\partial_2 = -p\partial_2\partial_1 \qquad \partial_1\partial_3 = -q\partial_3\partial_1\\
&\partial_2\partial_4 = -p\partial_4\partial_2 \qquad \partial_3\partial_4 = -q\partial_4\partial_3\\
&\partial_2\partial_2 = 0 = \partial_3\partial_3\\
&-(r^{-2}+r^2)\partial_1\partial_4 - \mathrm{i}r^2(r^{-2}-1)\partial_2\partial_3 + \mathrm{i}(r^{-2}-1)\partial_3\partial_2 + (r^{-2}+r^2)\partial_4\partial_1 = 0.
\end{aligned}
\tag{3.10}
$$

Even though $Q_1(\hat{R})$ is a hyperplane with anticommuting variables the differential calculus given by the $\hat{R}$ matrix (3.1) is quite peculiar. The reason for this may be traced to the fact that the root $-r^{-1}$ is multiple in nature and therefore the matrix $(E-B_1)\propto\Omega$ is nilpotent.

Previously we have seen how the $\hat{R}$ matrix formalism enables us to define the complete algebra of variables and derivatives (3.8), (3.9) and (3.10). The calculus is covariant with respect to the quantum supergroup $\mathrm{OSp}_{p,q}(2/2)$. The corresponding system of super-oscillators can be obtained by making the following substitutions:

$$
\begin{aligned}
&X^1 \to a_1^\dagger \qquad X^4 \to a_2^\dagger\\
&\partial_1 \to a_1 \qquad \partial_4 \to a_2\\
&X^2 \to b_1^\dagger \qquad X^3 \to b_2^\dagger\\
&\partial_2 \to b_1 \qquad \partial_3 \to b_2.
\end{aligned}
\tag{4.1}
$$

If we impose the reality condition (i.e. $\bar{p}=p$, $\bar{q}=q$) then Hermitian conjugation can be shown to be an inner antimultiplicative automorphism of the superoscillator algebra. It would have been more appropriate to use the generalization of the Wess–Zumino $\hat{R}$ matrix formalism to the supersymmetric case. However, the supersymmetric version does not modify the explicit algebra of coordinates and derivatives, and hence the super-oscillator algebra.

*PP would like to thank CSIR for financial support.*

## References

[1] Drinfield V G 1986 Quantum groups *Proc. Int. Congr. Math.* Berkeley vol 1 p 798
[2] Manin Yu I 1988 Quantum groups and noncommutative geometry *Preprint* Montreal University CRM-1561
[3] Faddeev L D, Rashetikhin N Yu, Takhtajan L A 1987 Quantization of Lie groups and Lie algebras *Preprint* LOMI E-14-87
Reshetikhin N Yu, Takhtajan L A and Faddeev L D 1990 *Leningrad Math. J.* **1** 193
[4] Chaichian M, Kulish P and Lukierski J 1991 *Phys. Lett.* **262B** 43
[5] Pusz W and Woronowicz S L 1989 *Rep. Math. Phys.* **27** 231
[6] Pusz W 1989 *Rep. Math. Phys.* **27** 349
[7] Wess J and Zumino B 1990 *Nucl. Phys.* B **18** 302
[8] Hlavaty L 1991 Yang–Baxter matrices and differential calculi on quantum hyperplanes Preprint PRA-HEP-91/1 and references therein
[9] Dabrowski L and Wang Lu-yu 1991 *Phys. Lett.* **266B** 51
[10] Schmidke W B, Vokos S P and Zumino B 1990 *Z. Phys.* C **48** 249
[11] Liao L and Song X-C 1991 *Mod. Phys. Lett.* A **6** 959
[12] Parashar P and Soni S K 1993 *J. Phys. A: Math Gen.* **26** 921